Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Vol. V. No. 6. March 12, 1908.

THE JOURNAL OF PHILOSOPHY
PSYCHOLOGY AND SCIENTIFIC METHODS

THE MEANING OF $\sqrt{-1}$

THE following discussion is an attempt to suggest something by way of interpretation of the square root of a negative quantity. It is, however, not the discussion of a mathematician. but, as will doubtless appear all too soon, of a layman in mathematics, who here and now makes apology to the mathematicians for presuming, unrobed and untrained, to stand in their pulpit. The layman must at times think and speak for himself, and though he may express only platitudes in a way as fantastic and unconventional as it is limited in point of view, his thinking and speaking are always of value to himself and are not always without some interest to others.

In general the contention here is that in the square root of a negative quantity there is—*except for an important qualification to be made hereafter*—only an application of the negative to the sphere or field of products or areas, that is, to a two-dimensional space, just as in the negative quantity, -1, there is such an application in the sphere of mere serial or linear composition or aggregation or, geometrically, of a one-dimensional space. So viewed, the imaginary quantity is essentially no more imaginary than the ordinary minus or negative quantity, although naturally the difference of the field produces different results and renders the quantity itself, $\sqrt{-1}$, useful in different ways.

The discussion proposed is best introduced by a simple problem whose solution will be seen at once to involve imaginary quantities. Thus, given a straight line, $2m$, the problem is to divide this into two parts so that the product of the parts will exceed m^2 by an amount n; and the solution is as follows: Let x be one of the parts. Then $2m - x$ is the other; $2mx - x^2 = m^2 + n$; and $x = m - \sqrt{-n}$, while the remaining part is $m + \sqrt{-n}$. In other words, as will hereafter be emphasized frequently, the excess of n over m^2, the product of the line's halves, takes the value of x, as well as that of $2m - x$, into the sphere of negative quantity, *but of negative quantity, not as length, but as the root or side of a negative area.* Were there no excess, were the desired product equal to, or less than,

m^2, the necessary parts would be quite without such complication.

Plainly, too, with regard to the values $m - \sqrt{-n}$ and $m + \sqrt{-n}$, no other result should be expected; for, in the first place, the equation $2mx - x^2 = m^2 + n$ is distinctly an equation of products or areas; in the second place, the problem, though dealing with a line, or a merely serial aggregate, distinctly asks for parts that are not just component lengths or parts, but sides of an area or factors of a product; and, in the third place, m^2 being the maximum or limiting product or area in the group of what may be called the normal or positive results of divisions of the aggregate or line, any product or area required to be in excess of m^2, if truly meeting the conditions of the problem, must encroach upon a field of minus areas. Similarly, in a purely linear field a line of length l could be so divided that one of its parts would exceed the line by s, but only if encroachment upon a field of negative lengths be allowed, the required parts being $l + s$ and $-s$.

That minus areas, or products, can have only dimensions, or factors, whose unit is $\sqrt{-1}$ is self-evident. Whence the meaning of this unit, or at least an important idea introductory to the meaning, must be, as asserted, that it represents nothing more nor less than an application of the negative to a two-dimensional field. In the solution of the given problem, however, the parts, sides or factors which were found, namely, $m - \sqrt{-n}$ and $m + \sqrt{-n}$, were only partially imaginary, and this only partial encroachment upon the field of negative areas seems to call for some explanation, although in the case of the line l, divided so that one part should exceed its length, the result was also a mixed one, the parts being $l + s$, or $l - (-s)$, and $-s$, or $l - (l + s)$, *and although also logically an unmixed negative is quite beyond what is thinkable.*

In explanation, then, of those mixed quantities $m + \sqrt{-n}$ and $m - \sqrt{-n}$, I venture simply to say that they are both mixed because they are both the sides or factors of a mixed area, or product, which is mixed for just the reason, already given, that m^2 is a maximum among all the possible wholly positive values resulting from division of $2m$. Surely, m^2 being a maximum, $m^2 + n$ can have no other meaning except that of the mixed area $m^2 - (-n)$, with the obvious factors $m + \sqrt{-n}$ and $m - \sqrt{-n}$; exactly as with the line l, where, l being the maximum, the part $l + s$ could have no other meaning save that of $l - (-s)$, and the part $-s$ no other meaning save that of $l - (l + s)$.

But now, with so much clear, it will greatly assist the present undertaking to broaden the view, for obviously the cases of division where the area, or product, exceeds m^2 are only terms in a series that will include not only these, but also such as are less than m^2 or as are greater than $2m^2$, and so on. Most generally, therefore, the terms in such a series may be put in the form $m^2 - (m - x)^2$, where $(m - x)^2$ expresses the difference between the maximum m^2 and the product, or area, that constitutes the term, and giving x definite values, the series runs: $m^2 - (m - 1)^2$, $m^2 - (m - 2)^2$, $m^2 - (m - 3)^2$, $\cdots$. But, so soon as x is larger than m, $(m - x)^2$ gets the value of $-n$,[1] and the terms will have the form, already recognized, of $m^2 - (-n)$, that is, $m^2 + n$, or, in definite values, $m^2 + 1$, $m^2 + 2$, $m^2 + 3$, $m^2 + 4$, $\cdots$. Furthermore, when n becomes equal to m^2, the subsequent terms will be $2m^2 + 1$, $2m^2 + 2$, $\cdots$. Yet, as to these last, it must be remembered that they also can have meaning only when understood as $m^2 - (-m^2 - 1)$, $m^2 - (-m^2 - 2)$ $\cdots$. And, not to go farther with this description of the series, it must be remembered also that *all the terms represent products or areas;* and it will be readily observed that as a whole the series is one of certain crises, or at least that the terms as formally represented suggest certain crises. When, for example, x becomes larger than m, the factors x and $2m - x$ at once become in part imaginary quantities, $m - \sqrt{-1}$ and $m + \sqrt{-1}$, $m + \sqrt{-2}$ and $m - \sqrt{-2}$, and so on; and, when n becomes equal to m^2, the term $m^2 - (-n)$, which *as in the series* will be $2m^2$, will have the *actual* value of zero, its factors being $m - m\sqrt{-1}$, and $m + m\sqrt{-1}$,—values which, as I understand them, indicate that m^2 has been taken twice, but in such a way that one m^2 belongs to the positive field, the other to the negative. Similarly in the case of the parts of the line l, if the excess of one part over the whole line were to be l, the excessive part *as a part* would be $2l$, or $l - (-l)$, and while, to repeat, such is in truth its length *as just such a part of the line,* its own independent value is zero, for it is itself composed of the two parts, $-l$ and $+l$. A diagram may be quite unnecessary, yet will do no harm. Thus, in Fig. 1, let AB be the line of length l, and S^1, S^2, S^3 be various dividing points, but S^l the point by which one part is made to exceed the whole line by its length l. Then S^lB *as a part of the line* is $2l$, the other part, S^lA, of course being $-l$, but *in itself,* that is, as itself a whole or a sum, S^lB is equal to zero. And in like manner, with apology for so much repetition, when n becomes equal to m^2, the resulting term is variously $2m^2$, $m^2 - (-m^2)$, and O.

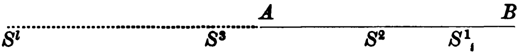

FIG. 1.

---

[1] With x larger than m, $(m - x)^2$ *quite by itself* would be $(-1)^2$, $(-2)^2$, $\cdots$, equal to $+1$, $+4$, $\cdots$, but *in the series under examination* these areas are made negative and the terms in which they belong, instead of being $m^2 - 1$, $m^2 - 2$, $\cdots$ are, as given, $m^2 + 1$, $m^2 + 2$, $\cdots$ or, again, $m^2 - (-1)$, $m^2 - (-2)$, $\cdots$.

The series, then, of the products, or rectangles, has these two crises; first, when $x < m$, and second, when $n = m^2$. There may be other crises, but another consideration now seems more urgent. The critical term, just shown to be equal to zero, is of course a product; it is a product with dimensions $m + m\sqrt{-1}$ and $m - m\sqrt{-1}$; and although a zero product, or area, having such factors may seem at first thought absurd, yet second thought will make it richly significant. At the very beginning of the series was there not also a product, or rectangle, equal to zero, but with dimensions $m + m$ and $m - m$? This initial term, it is true, could have only the area of the line $2m$, which though $2m$ in length would be zero in *actual* area, but such an area as this, though equal to zero, had a real meaning so soon as the problem, declaring the parts of the line to be factors of a product or sides of an area, was merely stated.[2] *The statement ipso facto transformed the mere line into a sphere of potential area, and the potentiality was actually in the zero whose factors,* or *dimensions, were* $m + m$ *and* $m - m$.[3] And the two rectangles, the two zero products or zero areas—what of them? What, in particular, of the peculiar difference between their factors, one having the factors $m + m\sqrt{-1}$ and $m - m\sqrt{-1}$, and the other, $m + m$ and $m - m$? The zero area of the imaginary factors, or dimensions, must also, as is plain, be a straight line of length $2m$, but can it be, following the suggestion of the symbols, that this line makes a right angle with the other? That in the series from one of these zeros to the other there is, geometrically described, a rotation, or at least, whatever be the intermediate transformation, a *virtual* rotation of the original line through 90°? That the zero area of the imaginary factors is not *a* second dimension, but a realization of *the* second dimension of the other zero area?

The questions are certainly important and they seem to call for an unequivocal answer in the affirmative. Lest, however, the suggestion of the symbols be not sufficiently clear, the following somewhat detailed geometrical demonstration is also given; and given only the more confidently since, besides representing the rotation, so-called, the various figures will also serve to clarify what has been said of the nature and order of the terms of the series, of the crises that characterize the series, and especially of the necessary encroachment of the series on a field of minus areas or products.

---

[2] The dimensions of the two zero areas, if written with full regard to their character or quality as not just flat quantities, but as dimensions, would be, in the first case, $m\sqrt{+1} + m\sqrt{-1}$ and $m\sqrt{+1} - m\sqrt{-1}$ and, in the second case, $m\sqrt{+1} + m\sqrt{+1}$ and $m\sqrt{+1} - m\sqrt{+1}$ or $2m\sqrt{+1}$ and $0\sqrt{+1}$.

[3] Or even this may be said: The zero area of a line has meaning in just so far as the line's division implies some functional relation, such as that of the sides of a rectangle, between the resulting parts or, arithmetically, as its recognized composition implies multiplication, that is, the treatment of the parts, not merely as component parts, but also as factors. Just such an implication makes the line an area *potentially;* nor is such potentiality only a polite name for unreality.

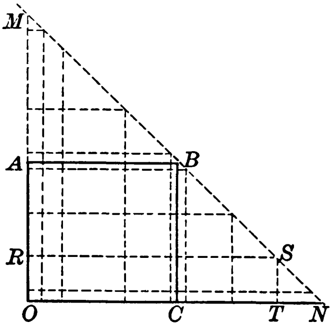

FIG. 2.

Thus, in each of the accompanying figures, *ORST* represents some term in the series. Fig. 2 gives an indefinite number of terms, or rectangles, and shows how a rectangle, as it were, emerges out of the line *ON, ON* being equal to $2m$; gradually moves, in sides and area, towards *OABC,* which is m^2; becomes equal to, and identical with, *OABC*; and eventually, leaving *OABC,* loses itself in the line *OM* perpendicular to *ON* and also equal in length to it. In this figure, too, as also in the other figures, *TN* and *OT* are x and $2m - x$, respectively. In Figs. 3–8 the square *HIJB,* representing the difference between m^2 and $2mm - x^2$, that is, between the *actual* maximum and the general term of the series, first, in Fig. 5 is seen, after a gradation from an equivalence with this maximum, to have become zero; secondly, in Fig. 6 and thereafter, *having passed through zero,* to be a minus area; and, thirdly, in Fig. 8, to be again equivalent to, or identical with, *OABC,* the maximum. And in Figs. 6 and 7, indeed also in Fig. 8, the dimensions of the general term *ORST,* always including as these do the side or root of the minus area *HIJB,* must be in part "imaginary." *RS* and *ST,* I say, must be "imaginary" lines; they are imaginary to just the extent of their including, the one positively and the other negatively, the lines *HI* and *IJ,* since either *HI* or *IJ* is equal to $\sqrt{-HIJB}$;[4]

---

[4] So do we see how a line as just a line and a line as in a given relation, as the side of an area, for example, and of an area, too, that is itself in a given relation to other areas, are of different values. As just a line *HI* is just *HI,* having some given length. As just a side of the square *HIJB,* it is indeed not merely *HI,* but still merely $\sqrt{HIJB}$. As the side, however, of *HIJB in the figures* 6–8, representing a portion of the series now in mind, it is $\sqrt{-HIJB}$.

and, again, they are "imaginary," although when capable of such simple geometrical demonstration they lose much of their supposedly imaginary character.

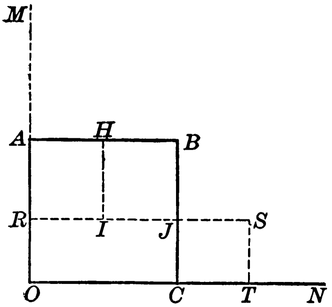

Fig. 3.

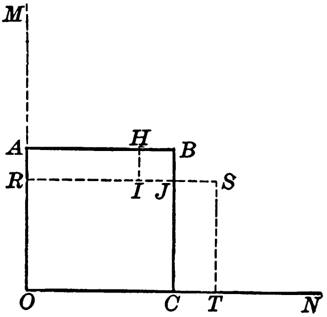

Fig. 4.

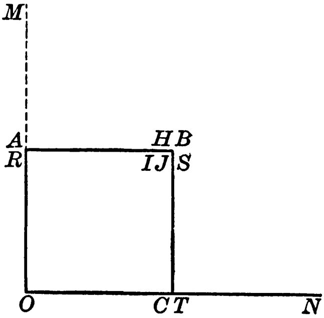

Fig. 5.

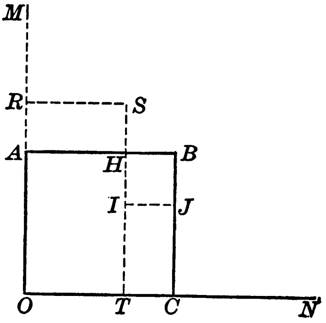

Fig. 6.

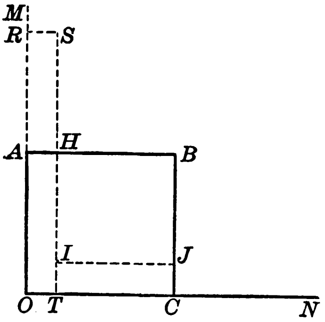

Fig. 7.

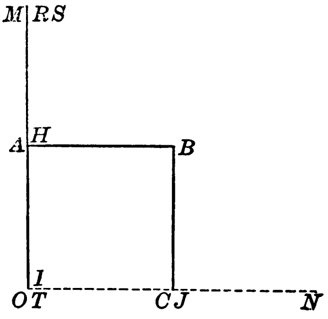

Fig. 8.

And so, though with regard to but one simple, specific case, I would suggest what is, or what may be, the meaning of $\sqrt{-1}$. Certainly it seems, on the evidence before us, that, if meanings have parts, at least in part $\sqrt{-1}$ means the application of the negative, or, more specifically, of the idea of the negative quantity, to the sphere or field of products or areas, and, as will be recalled, this is the notion of it, subject to a promised qualification, asserted at the very beginning of the present discussion. Yet, before the important qualification is made, there are several minor suggestions, more or less closely connected with what has been said, to which brief attention is asked.

1. Not, indeed, the actual or static or positive value of *OM* or of the product whose factors are $m + m\sqrt{-1}$ and $m - m\sqrt{-1}$, but the *realized* value, the value *as at a certain point in the series,* is $2m^2$, and $2m^2$, it is to be specially observed, *is equal to the total field or area of the series from ON to OM.*

2. The rectangle *ORST* as required to be greater than the maximum square *OABC,* not only is what gives rise to the imaginary quantity, but also, being statically or constructively impossible, calls for just such a generated area as is so conspicuously evident in the term *OM* or product $2m^2$ and as also in less degree is to be seen in every other term of the series. Such generation of area is, of course, closely connected with the meaning of the rotation that is effected by the series and that has been indicated already.

3. The values of x and $2m - x$, always having the forms $m - \sqrt{n}$ and $m + \sqrt{n}$ or $m - \sqrt{-n}$ and $m + \sqrt{-n}$, are such as to suggest a straight line, the equation for which, *MO* and *ON* being the coordinates, would be $x + y = 2m$. This equation, however, would only hide the relation of the line to the series. This line is *MN* (Fig. 2), which is (1) the path of the dividing point *S,* whose divisions on *MN* are proportional to those of *T* on *ON,* and (2), as the path of such proportional divisions and *as a diagonal,* a projection of the line *ON* on the peculiar plane or two-dimensional field of the series. *MN,* in fact, when regarded in the light of its place in the series, or in the geometrical representation of the series, is what I should like to call a two-dimensional line and the points in its path would be two-dimensional points or area-points, some of them, as in the part *MB,* being points partly in the negative and partly in the positive field or being points in what above was styled a mixed area—all of which, perhaps, is inexcusably subtle, but it seems to me to be not without some point of its own! Those who still prefer the equation $x + y = 2m$, may fall back upon that!

But, finally, the promised qualification, in effect only a recognition of the limited view afforded by just one simple case, is that the notion of $\sqrt{-1}$ here asserted, like any other notion, can be only partially or relatively true. Inasmuch, however, as the relatively true is always bound to be informed with, or to have some contact with, what is really and fully true, there is always ground for generalization even from what is relative, and being a philosopher, at least by profession, I find the temptation to generalize irresistible. Perhaps, moreover, generalization is only the proper complement, at least poetically just, of qualification. Does it not quickly transform that from which it proceeds?

The notion of the product or area, so essential to the foregoing study, is certainly a wider and deeper one than its place in the distinction between a one-dimensional and a two-dimensional field, or between mere linear aggregation or addition and multiplication, would make it. Or at least this: So far as the notion of area has been used here, it is really more general, being applicable to any definite portion of an $n+1$-dimensional field relatively to an n-dimensional field. Indeed, in the constant association of areas with products, that must have been conspicuous in the earlier paragraphs of this paper, such a general view of areas was already implied. A cube, or any solid, is a product—the area of an area, and so on.

Now the particular problem examined above was a problem that to all intents and purposes was solved only by the evolution of a two-dimensional field out of a one-dimensional field, and the maximum, the static or so-called *actual* maximum in the case, representing nothing more nor less than the peculiar limitation of the one-dimensional field in a form relative to the particular problem, forced upon the process of the evolution, or rather upon the formal description of this process, the imaginary quantity. This quantity was negative because of the limiting maximum, and "imaginary" because the difference in the dimensions demanded a transcendence of the formal possibilities of the one-dimensional field. As to this transcendence, it was especially apparent at two, nay, really at three, places in the series,[5] and the part of the imaginary quantity in the realization of it was also evident. Of course, specifically, the $\sqrt{-1}$ was only the unital factor of a minus product, or area, just as the primrose is said to have been to a certain person only a primrose, but as just such a factor the $\sqrt{-1}$, or the negative area, whose sides it measured, was also a bridge, real and serviceable, between the one-dimensional and the two-dimensional fields—overcoming the limitations of the former, yet serving as a means also to the realization of the latter; and bridging those fields was the very crux of the problem.

And in the specific problem treated one thing more must be remarked, though this may be only a rewording of something already said. Associated with the imaginary quantity and plainly essential to its meaning and value in the series, there was the idea of movement, if not also of something very like real and effective action. There was, for example, absolutely necessary to the interpretation of such a value as $2m^2$ or as that of any term in excess of m^2, the distinction between the formally actual and the realized value of the term. An important element, therefore, in the meaning of the imaginary quantity, appears to be, not merely the transition from one dimension to two, as already pointed out, or the formal transcendence of certain limitations, but something more distinctly dynamic than that. At risk of seeming only poetic, I would even say that, having terms with *realized* values so different from their formally actual values, the series seems to *do* something; it makes or achieves, or, as was said before, it generates area, not merely has or finds area.

---

[5] At the first term potentially already an area, though in area actually zero; at the term in which the difference between the maximum and the actual term became zero; and at the term that was $2m^2$, or $m^2-(-m^2)$, or 0.

But the notion of area is a wide one, and, now to generalize from the conditions of the particular problem, is not the essence of any real problem, mathematically put, but just the bridging of spheres of different dimensions, or say, more simply, of n dimensions and $n+1$ dimensions? Is it not the realization of area for an n-dimensional field?[6] Every n-dimensional field, however, must have its own natural maxima or minima or, more generally, its own peculiar limits, and any problem that may be formulated consistently with the structure of this field must not only encounter these limits, but also, as it is solved, must transcend them, and, mathematically, the negative quantity is often the means for the transcendence. Moreover, where there is a negative quantity there must be also an imaginary quantity of some sort and degree. In this *general* imaginary quantity, as in the $m + \sqrt{-n}$ of the problem here discussed, the relations of the n-space are seen to take their place, by a sort of projection, in the transcendent or negative sphere of the $n+1$ space. Nor is this *general* imaginary quantity necessarily a formally impossible one, like $m + \sqrt{-n}$; it is the formally impossible result of any operation, *the only condition apparently being that the operation determining the imaginary term must be the reversal of the operation by which the generation of the new sphere beyond its peculiar limits has taken place.* Thus, again, in the special case, the generated area was one of rectangles or products; the imaginary quantity was a side or root or factor.

---

[6] Otherwise put, the typical problem involves a functional treatment of some given field or structure, such as, for a very simple example, the even or the proportional division of a line. Functions, however, are always more general than the fields or structures within which they are applied. And so, as they are developed, they always require transcendence of the formal possibilities of these fields, that is, a change in the number of dimensions. The infinite series implies such a change or such transcendence and, as here shown, the imaginary quantity is almost an open expression of it.

The line l, too, recalled from above, affords an illustration, with which I may conclude, of the very general idea here in mind, for the notion of dimensions must also be capable of very wide generalization. Although merely as regards direction the straight line is in a one-dimensional field, yet this very thing, the direction or straightness, relatively to the *defined* length, constitutes a second dimension, the given defined length being a first dimension, and the problem of division, calling as it did for a transcendence of this first dimension, could be solved only by the use of the negative quantity, which at once involved the imaginaries, $-s$ and $l-(-s)$, that plainly were parts, not by composition or inclusion, but by the reverse operation of exclusion.

ALFRED H. LLOYD.

UNIVERSITY OF MICHIGAN.

DISCUSSION

THE PROBLEM OF OBJECTIVITY

THE task which contemporary realism regards as its special mission is to protect the objects of the physical world against the encroachments of consciousness. Anxious to avoid the error of the older realism, it has repeatedly disavowed all subjectivistic proclivities and asserted that consciousness is merely a relation among the objects and qualities which it knows. Qualities or "contents" which earlier realism regarded as purely mental are now transferred to the other side of the line and take their place alongside of the "primary qualities" as constituent members of the "objective" world.

While this spoliation of consciousness is going on, the onlooker naturally attempts to forecast the final result. So far as the data enable him to determine, there can be but one issue to the process. If the taint of subjectivism is to be completely eliminated, all contents whatsoever must be stripped away. So long as any content is left, whether it be much or little, earlier realism finds a sanction for its assumption that perceived qualities can be constituted by consciousness, and the innovations of its successor become essentially matters of detail.

If, however, we remove all contents as indicated, a situation is created which appears to involve serious difficulties. All diversity being lodged in the object, consciousness necessarily becomes qualitatively identical in all its manifestations. There can be no different forms of consciousness, but only consciousness of different objects. Having approximated the vanishing-point of abstraction, consciousness is reduced to a single dimension, that of colorless or bare awareness.